Journal of Academia and Industrial Research (JAIR)
Volume 2, Issue 2 July 2013

ISSN: 2278-5213

RESEARCH ARTICLE

$\overline{X}$ Charts with Variable Sampling interval and Warning limits

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Abstract

The idea of *variable sampling interval and warning limits* (VSIWL) is proposed for $\overline{X}$ charts. Expressions for the performance measures for the charts with VSIWL are developed. The methods presented are general and can be applied to other Shewhart control charts. The performances of VSIWL $\overline{X}$ charts are compared numerically with that of VSI $\overline{X}$ charts with and without runs rules for switching between sampling interval lengths. It is observed that in general the former charts perform significantly better than the later.

Keywords: Adaptive control chart, average number of samples to signal, statistical process control, Shewhart control charts.

Introduction

Nowadays it has been well recognized that adaptive control charts are significantly more efficient than the static ones. Reynolds *et al*. (1988) were the first to consider the intuitive notion of adapting sampling interval length of a control charts according to the status of a process indicated by the last plotted sample point. They proposed *variable sampling interval* (VSI) $\overline{X}$ charts. The principle of choosing the sampling interval length in a VSI chart is that as the location of the current sample point approaches the control limits, tighten the control by taking the next sample more quickly. The in-control area of the chart is partitioned into a central region and one or more warning regions. Each region determines length of the sampling interval for the next sample if the current sample point falls in it. Reynolds *et al.* (1988) showed that the idea of VSI substantially improves the statistical performance of $\overline{X}$ charts. Also, they showed that the statistical performance of a VSI $\overline{X}$ chart in detecting a shift of any magnitude that exists initially is optimized by using the dual sampling interval policy consisting of the shortest and longest possible sampling interval lengths. Afterwards, Prabhu*et al.* (1993) and Costa (1994) independently proposed variable sample size $\overline{X}$ charts. Prabhu *et al.* (1994) proposed variable sample size and sampling interval $\overline{X}$ charts. Costa (1999a) proposed the adaptive $\overline{X}$ charts in which all the three design parameters are variable. Tagaras (1998) reported an extensive survey of the research on adaptive control charts until 1997. Then also, various schemes of adaptive control charts have been proposed and extensively investigated with different perspectives. See for example, Amin and Widmaier (1999), Costa(1999b), Aparasi and Haro (2003), Costa and Rahim (2001), Epprecht *et al.* (2003), Zimmer *et al.* (1998), Reynolds and Stoumbos (2001), Wu *et al.* (2005), Yu and Hou (2006), Celano *et al.* (2006), Chen (2007), Wu *et al.* (2007), Yang and Su (2007), Mahadik and Shirke (2007a, b), Jiang *et al.* (2008), Jensen *et al.* (2008), Luo *et al.* (2009), Wu *et al.* (2009), Shi *et al.* (2009), De Magalhaes *et al.* (2009), Celano (2009), Faraz and Moghadam (2009), Mahadik and Shirke (2009, 2011), Li and Wang (2010), Epprecht *et al.* (2010), Shu *et al.* (2010), Mahadik (2012a, b, 2013), Chen *et al.* (2011), Dai *et al.* (2011),Faraz and Saniga (2011), Nenes (2011), Kooli and Limam (2011) and Lee (2011).

The weakness of any adaptive control chart is the inconvenience in its administration due to the frequent switches between the values of its adaptive design parameters. In order to reduce the frequency of switches between sampling interval lengths of VSI charts, Amin and Letsinger (1991) proposed the use runs rules for switching between these lengths. Amin and Hemasinha (1993) developed approximate expressions for the performance measures for VSI $\overline{X}$ charts with such runs rules while Mahadik (2011a) developed the exact expressions.

In the present study, the idea of variable warning limits is proposed for VSI $\overline{X}$ charts. This significantly improves statistical performances of the charts in detecting small to moderate shifts in the process mean and also dramatically reduces the frequency of switches between sampling interval lengths.

Materials and methods

A VSIWL $\overline{X}$ *Chart:* Let the quality characteristic X to be monitored follows a normal distribution with mean μ, and a known and constant standard deviation σ. Suppose μ_0 is the target value of μ.

An occurrence of an assignable cause results in a shift of size δ in μ, where δ is expressed in σ units. It is assumed that δ remains constant following the occurrence of a shift until it is detected. A VSIWL $\bar{X}$ chart to monitor μ is as described below.

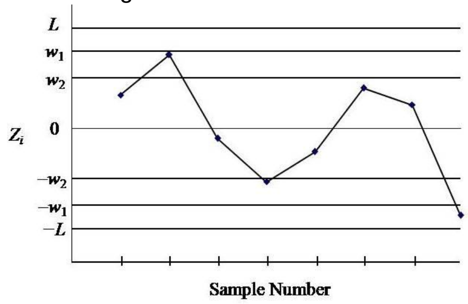
The chart statistic is the standardized sample mean $Z_i = \sqrt{n}\left(\bar{X}_i - \mu_0\right)/\sigma$, where $\bar{X}_i$, i = 1, 2, …, is the mean of i^{th} sample of size is n drawn on X. Note that when $\mu = \mu_0$, $Z_i \sim N$ (0, 1), and when $\mu = \mu_0 + \delta\sigma$, $Z_i \sim N$ ($\sqrt{n}\delta$, 1). Each control limit of the chart is at the distance of L units from its centerline. Let $t(i)$ be the length of sampling interval between $(i - 1)^{st}$ and i^{th} trials and $w(i)$ be the distance of each warning limit from the centerline for the i^{th} trial, i = 1, 2, …. The values of $(t(i), w(i))$ can be either (t_1, w_1) or (t_2, w_2), where t_1, t_2, w_1, and w_2 are such that $t_{\max} \geq t_1 \geq t_2 \geq t_{\min}$ and $L > w_1 \geq w_2 > 0$, where $t_{\min}$ and $t_{\max}$ are the shortest and longest possible sampling intervals, respectively. When Z_{i-1} falls within the control limits, the pair of values of $(t(i), w(i))$, i = 2, 3, …,between (t_1, w_1) and (t_2, w_2) is chosen according to the following rule

$$(t(i), w(i)) = \begin{cases} (t_1, w_1), & \text{if } Z_{i-1} \in I_1 \\ (t_2, w_2), & \text{if } Z_{i-1} \in I_2, \end{cases}$$

where $I_1 = [-w(i), w(i)]$ and $I_2 = (-L, -w(i)) \cup (w(i), L)$ for the i^{th} trial, i = 1, 2, …

At start-up the values of $(t(1), w(1))$ can be chosen using an arbitrary probability distribution, as no prior sample is available. In practice, it is recommended to use the pair (t_2, w_2) for the first trial to provide additional protection against the problems that may exist initially. The trial following an out-of-control signal is again treated to be the first trial and the mechanism of choosing $(t(i), w(i))$is restarted from that. The chart signals an out-of-control state when a sample point falls beyond the control limits. Figure 1 shows a typical VSIWL $\bar{X}$ chart.

Fig. 1. A VSIWL $\bar{X}$ chart.



In practice, only one set of the warning limits may be shown anywhere on the chart within the control limits to represent the two sets in order to avoid the complexity in the administration. Suppose each warning limit of this set is at a distance of w units from the centerline. When $w(i) = w_j$, j = 1, 2, plot Z_i anywhere within $[-w, w]$,$(-L, -w)$, and (w, L), respectively, when it is within $[-w_j, w_j]$, $(-L, -w_j)$, and (w_j, L). Note that when $w_1 = w_2$, a VSIWL $\bar{X}$ chart is a VSI $\bar{X}$ chart. In the next section, expressions for performance measures for a VSIWL $\bar{X}$ chart are derived.

Performance measures: The appropriate measures of statistical performance of a VSIWL $\bar{X}$ chart are the steady-state average time to signal (SSATS) and the average number of samples to signal (ANSS). SSATS is the expected value of the time between a shift that occurs at some random time after the process starts and the time the chart signals while ANSS is the expected value of the number of samples taken from a shift to the time the chart signals. The administrative performance can be measured through average number of switches to signal (ANSW). ANSW is the expected value of the number of switches between two sampling interval lengths from a shift to the signal.

Let $SSATS_\delta$, $ANSS_\delta$, and $ANSW_\delta$ be the SSATS, ANSS, and ANSW, respectively of a control chart when the process mean has shifted from μ_0 to $\mu_1 = \mu_0 + \delta\sigma$. In the following, first the expressions for $SSATS_\delta$ and $ANSS_\delta$ are derived using a Markov chain approach. Brook and Evans (1972) were the first to use this approach to find the average run length of a control chart. Henceforth, the i^{th} trial refers to the i^{th} trial after a shift when $i > 0$ and the last trial before the $(i + 1)^{st}$ trial when $i \leq 0$.Also, Z_i refers to the sample point corresponding to the i^{th} trial.

Define the three states 1, 2, and 3 of the Markov Chain corresponding to whether a sample point is plotted in I_1, I_2 and $I_3 = (-\infty, -L] \cup [L, \infty)$, respectively. State 3 is the absorbing state, as the process of taking samples is restarted when a sample point falls in region I_3. The transition probability matrix is given by

$$\mathbf{P}^\delta = \begin{bmatrix} p_{11}^\delta & p_{12}^\delta & p_{13}^\delta \\ p_{21}^\delta & p_{22}^\delta & p_{23}^\delta \\ 0 & 0 & 1 \end{bmatrix},$$

Where p_{jk}^δ is the transition probability that j is the prior state and k is the current state, when the process mean has shifted by $\delta\sigma$.

For example,

$$p_{12}^{\delta} = \Pr_{\delta}[Z_i \in I_2 |\ Z_{i-1} \in I_1]$$
$$= \Pr_{\delta}[Z_i \in I_2 | w(i) = w_1]$$
$$= \Pr_{\delta}[-L< Z_i < -w_1] + P[w_1 < Z_i < L]$$
$$= \Phi(-w_1 - \sqrt{n}\delta) - \Phi(-L - \sqrt{n}\delta) + \Phi(L - \sqrt{n}\delta) - \Phi(w_1 - \sqrt{n}\delta),$$

Where $\Phi(\cdot)$ is the cumulative distribution function of standard normal variate.

Then, SSATS$_\delta$ and ANSS$_\delta$ are given by

SSATS$_\delta$ = $\boldsymbol{b}'(\mathbf{I}-\mathbf{P}_1^{\delta})^{-1}\boldsymbol{t}$−E($U$) (1)

And ANSS$_\delta$ = $\boldsymbol{b}'(\mathbf{I}-\mathbf{P}_1^{\delta})^{-1}\mathbf{1}$,

Where **I** is the identity matrix of order 2, $\mathbf{P}_1^{\delta}$ is the sub matrix of $\mathbf{P}^{\delta}$ that contains the probabilities associated with the transient states only, $\boldsymbol{t}' = (t_1, t_2)$, $\mathbf{1}'$ = (1, 1), and $\boldsymbol{b}' = (b_1, b_2)$, b_j being the conditional probability that Z_0 falls in I_j given that it falls within the control limits, j = 1, 2. We note that $b_2 = 1 - b_1$. The Expression for b_1 is derived in appendix A.

E(U) in equation (1) is the expected value of the time U between the 0th trial and the shift. Assuming that an assignable cause of a process shift occurs according to a Poisson process, it can be shown that E(U) = $\mathrm{E}[t(1)]/2$.

Hence, SSATS$_\delta$ = $\boldsymbol{b}'(\mathbf{I}-\mathbf{P}_1^{\delta})^{-1}\boldsymbol{t} - \mathrm{E}[t(1)]/2$.

Now, to derive the expression for ANSW$_\delta$, let O_i be the number of switches between two sampling interval lengths following the i^{th} trial until the signal, i = 1, 2, … Further, let

$o_{j.i}^{\delta} = \mathrm{E}_{\delta}(O_i | Z_{i-1} \in I_j)$, i = 1, 2, … ,j = 1, 2.

Then, the expression for ANSW$_\delta$ is given by

ANSW$_\delta$ = $\mathrm{E}_{\delta}[O_1] = b_1 o_{1.1}^{\delta} + b_2 o_{2.1}^{\delta} = \boldsymbol{b}'\mathbf{O}_1^{\delta}$,

Where, $\mathbf{O}_s^{\delta} = (o_{1.s}^{\delta}, o_{2.s}^{\delta})'$, s = 1, 2, …

The expression for $\mathbf{O}_1^{\delta}$ is derived in appendix B.

Alternatively, the expression for ANSW$_\delta$ can also be obtained using the Markov Chain approach. For, let

$$Y_i = \begin{cases} 1, \text{ if } (Z_{i-1} \in I_1, Z_i \in I_2) \\ 2, \text{ if } (Z_{i-1} \in I_2, Z_i \in I_1) \\ 3, \text{ if } (Z_{i-1} \in I_1, Z_i \in I_1) \\ 4, \text{ if } (Z_{i-1} \in I_2, Z_i \in I_2) \\ 5, \text{ if } |Z_i| > L \end{cases}, i = 1, 2, \ldots$$

It is easy to see that $\{Y_i$, i = 1, 2, …} is a Markov Chain with transition probability matrix

$$\mathbf{Q}^{\delta} = \begin{bmatrix} 0 & p_{21}^{\delta} & 0 & p_{22}^{\delta} & p_{23}^{\delta} \\ p_{12}^{\delta} & 0 & p_{11}^{\delta} & 0 & p_{13}^{\delta} \\ p_{12}^{\delta} & 0 & p_{11}^{\delta} & 0 & p_{13}^{\delta} \\ 0 & p_{21}^{\delta} & 0 & p_{22}^{\delta} & p_{23}^{\delta} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the expression for ANSW$_\delta$ is given by

ANSW$_\delta$ = $\boldsymbol{a}'(\mathbf{I}_1 - \mathbf{Q}_1^{\delta})^{-1}\boldsymbol{e}$,

Where, $\mathbf{I}_1$ is the identity matrix of order 4, $\mathbf{Q}_1^{\delta}$ is the sub matrix of $\mathbf{Q}^{\delta}$ that contains the probabilities associated with the transient states only, $\boldsymbol{e} = (1, 1, 0, 0)'$, and $\boldsymbol{a} = (a_1, a_2, a_3, a_4)'$, a_j being the initial probability of state j, j = 1, 2, 3, 4, given by

$$a_j = \Pr_{\delta}[Y_1 = j] = \begin{cases} b_1 p_{12}^{\delta}, & j = 1 \\ b_2 p_{21}^{\delta}, & j = 2 \\ b_1 p_{11}^{\delta}, & j = 3 \\ b_2 p_{22}^{\delta}, & j = 4 \end{cases}.$$

Results and discussion

Performance evaluation of VSIWL $\bar{X}$ charts: The performances of VSIWL $\bar{X}$ charts are evaluated by comparing that with that of VSI, VSI (1, 3), and VSI (2, 3) $\bar{X}$ charts, where VSI (k, m) $\bar{X}$ charts refers to the VSI $\bar{X}$ charts with runs rule (k, m) for switching between sampling interval lengths. When the successive m sample points before the i^{th} trial fall within the control limits, rule (k, m) chooses sampling interval length t_1 for the i^{th} trial if among those m sample points, the number of sample points falling in each warning region is less than k, otherwise it chooses the sampling interval length t_2. See Mahadik (2011a) for the details of VSI (k, m) $\bar{X}$ charts.

Among various runs rules considered by Mahadik (2011a), runs rule (1, 3) reduces the ANSW values of VSI $\bar{X}$ charts the most without affecting their SSATS values for small to large shifts in the process mean. Further, runs rule (2, 3) significantly reduces both, the ANSW values for shifts of all sizes and SSATS values for small shifts without affecting that for large shifts. Hence, the VSI $\bar{X}$ charts with these runs rules are chosen for comparison.

Table 1. The SSATS values of the matched VSIWL, VSI, VSI (1, 3), and VSI (2, 3) $\overline{X}$ charts.

Chart	w_1	w_2	SSATS for a shift of size									
			0	0.25σ	0.5σ	0.75σ	1σ	1.5σ	2σ	2.5σ	3σ	4σ
Case 1: n= 1, t_1 = 2, t_2 = 0.2, L = 3												
VSIWL	1.40	0.16	369.90	267.80	129.49	55.91	24.35	5.79	2.10	1.11	0.75	0.54
VSIWL	2.20	0.03	369.90	259.80	118.03	49.16	22.06	6.42	2.72	1.45	0.92	0.57
VSI	0.59		369.90	274.69	142.14	66.98	31.51	7.67	2.40	1.11	0.73	0.54
VSI (1, 3)	1.18		369.90	268.63	130.38	55.85	23.65	5.23	1.93	1.06	0.74	0.54
VSI (2, 3)	0.39		369.89	267.35	128.13	54.04	22.70	5.30	2.23	1.37	0.98	0.63
Case 2: n = 2, t_1 = 1.8, t_2 = 0.4, L = 3												
VSIWL	1.30	0.18	369.90	209.38	71.76	24.76	9.78	2.49	1.07	0.68	0.55	0.50
VSIWL	2.00	0.04	369.90	201.66	65.60	22.89	9.66	2.74	1.18	0.71	0.56	0.50
VSI	0.56		369.90	216.33	79.68	29.08	11.50	2.62	1.06	0.67	0.55	0.50
VSI (1, 3)	1.16		369.90	209.20	70.82	23.72	9.17	2.40	1.07	0.68	0.55	0.50
VSI (2, 3)	0.38		369.89	207.90	69.51	23.23	9.18	2.64	1.27	0.79	0.59	0.50
Case 3: n = 3, t_1 = 1.2, t_2 = 0.6, L = 3												
VSIWL	1.60	0.27	369.90	175.09	50.91	16.48	6.50	1.73	0.80	0.56	0.51	0.50
VSIWL	2.10	0.08	369.90	172.12	49.12	16.25	6.67	1.83	0.82	0.57	0.51	0.50
VSI	0.96		369.90	179.09	54.71	18.20	7.03	1.73	0.79	0.56	0.51	0.50
VSI (1, 3)	1.52		369.90	175.73	51.36	16.53	6.46	1.72	0.80	0.56	0.51	0.50
VSI (2, 3)	0.64		369.90	173.64	49.67	15.97	6.42	1.86	0.88	0.59	0.51	0.50
Case 4: n = 4, t_1 = 1.5, t_2 = 0.5, L = 3												
VSIWL	1.40	0.20	369.90	140.23	32.02	9.14	3.51	1.03	0.60	0.51	0.50	0.50
VSIWL	1.80	0.09	369.90	136.30	30.59	9.05	3.60	1.07	0.60	0.51	0.50	0.50
VSI	0.67		369.90	147.14	36.19	10.31	3.71	1.02	0.59	0.51	0.50	0.50
VSI (1, 3)	1.26		369.90	140.48	31.60	8.83	3.41	1.03	0.60	0.51	0.50	0.50
VSI (2, 3)	0.45		369.92	138.65	30.74	8.80	3.58	1.18	0.65	0.52	0.50	0.50
Case 5: n = 5, t_1 = 1.4, t_2 = 0.3, L = 3												
VSIWL	1.50	0.29	369.90	115.18	20.52	4.94	1.86	0.69	0.52	0.50	0.50	0.50
VSIWL	1.90	0.12	369.90	110.48	19.12	4.98	2.00	0.73	0.53	0.50	0.50	0.50
VSI	0.91		369.90	122.74	24.70	5.98	2.00	0.68	0.52	0.50	0.50	0.50
VSI (1, 3)	1.47		369.90	115.52	20.38	4.78	1.83	0.69	0.52	0.50	0.50	0.50
VSI (2, 3)	0.60		369.91	111.77	18.84	4.71	2.05	0.85	0.56	0.50	0.50	0.50

Table 2. The ANSW values of the matched VSIWL, VSI, VSI (1, 3), and VSI (2, 3) $\overline{X}$ charts.

Chart	w_1	w_2	ANSW for a shift of size									
			0	0.25σ	0.5σ	0.75σ	1σ	1.5σ	2σ	2.5σ	3σ	4σ
Case 1: n = 1, t_1 = 2, t_2 = 0.2, L = 3												
VSIWL	1.40	0.16	52.29	40.26	22.60	11.49	5.65	1.42	0.52	0.31	0.21	0.07
VSIWL	2.20	0.03	8.27	6.57	3.91	2.12	1.15	0.47	0.30	0.23	0.16	0.06
VSI	0.59		182.42	137.53	73.98	36.49	17.74	4.18	1.10	0.43	0.24	0.07
VSI (1, 3)	1.18		77.77	59.09	32.00	15.38	6.95	1.37	0.49	0.32	0.22	0.07
VSI (2, 3)	0.39		108.88	80.05	39.82	17.18	7.00	1.31	0.55	0.36	0.22	0.05
Case 2: n = 2, t_1 = 1.8, t_2 = 0.4, L = 3												
VSIWL	1.30	0.18	60.61	37.18	14.66	5.32	1.95	0.45	0.24	0.12	0.05	0.00
VSIWL	2.00	0.04	13.59	8.72	3.66	1.46	0.67	0.31	0.20	0.11	0.04	0.00
VSI	0.56		180.93	107.51	40.55	14.51	5.17	0.80	0.27	0.13	0.05	0.00
VSI (1, 3)	1.16		77.91	46.75	17.25	5.51	1.70	0.41	0.25	0.13	0.05	0.00
VSI (2, 3)	0.38		108.28	61.04	19.55	5.44	1.62	0.47	0.26	0.11	0.03	0.00
Case 3: n = 3, t_1 = 1.2, t_2 = 0.6, L = 3												
VSIWL	1.60	0.27	52.79	29.67	10.84	3.54	1.25	0.42	0.20	0.06	0.01	0.00
VSIWL	2.10	0.08	16.31	9.82	3.82	1.40	0.67	0.34	0.17	0.05	0.01	0.00
VSI	0.96		164.18	85.94	29.82	10.12	3.42	0.60	0.22	0.06	0.01	0.00
VSI (1, 3)	1.52		62.27	34.70	12.58	3.97	1.29	0.43	0.21	0.06	0.01	0.00
VSI (2, 3)	0.64		95.60	48.90	14.50	3.70	1.16	0.42	0.15	0.03	0.00	0.00
Case 4: n = 4, t_1 = 1.5, t_2 = 0.5, L = 3												
VSIWL	1.40	0.20	58.82	25.95	6.72	1.70	0.60	0.24	0.08	0.01	0.00	0.00
VSIWL	1.80	0.09	25.62	11.85	3.21	0.94	0.45	0.22	0.07	0.01	0.00	0.00
VSI	0.67		184.70	76.37	19.08	4.68	1.26	0.27	0.08	0.01	0.00	0.00
VSI (1, 3)	1.26		76.21	32.51	7.60	1.59	0.55	0.25	0.08	0.01	0.00	0.00
VSI (2, 3)	0.45		109.31	41.28	7.69	1.48	0.60	0.24	0.06	0.01	0.00	0.00
Case 5: n = 5, t_1 = 1.4, t_2 = 0.3, L = 3												
VSIWL	1.50	0.29	61.72	25.44	6.18	1.47	0.58	0.22	0.04	0.00	0.00	0.00
VSIWL	1.90	0.12	25.80	11.40	2.91	0.86	0.46	0.20	0.04	0.00	0.00	0.00
VSI	0.91		170.96	64.84	15.67	3.72	1.03	0.24	0.04	0.00	0.00	0.00
VSI (1, 3)	1.47		65.75	26.94	6.34	1.37	0.55	0.22	0.04	0.00	0.00	0.00
VSI (2, 3)	0.60		99.72	35.45	6.31	1.23	0.57	0.17	0.02	0.00	0.00	0.00

The four charts mentioned above are designed such that their in-control statistical performances are matched. This is done by keeping the design parameters *n*, t_1, t_2, and *L* of all the charts the same and choosing the warning limits of each chart such that E[*t*(1)] = t_0 holds for each chart, where t_0 is some suitable constant.

As a VSIWL $\bar{X}$ chart has two sets of warning limits, by fixing one of them this condition uniquely determines the other. By fixing w_1, we get

$$w_2 = \Phi^{-1}\left\{\frac{(t_0 - t_2)[2\Phi(L) - 1 - 2\Phi(w_1)] + t_1 - t_2}{2(t_1 - t_0)}\right\},$$

or by fixing w_2, we get

$$w_1 = \Phi^{-1}\left\{\Phi(L) + \frac{[2\Phi(w_2) - 1](t_0 - t_1)}{2(t_0 - t_2)}\right\}.$$

In the same way the warning limits of VSI, VSI (1, 3), and VSI (2, 3) $\bar{X}$ charts are determined.

The SSATS and ANSW values of such statistically matched charts are then computed for shifts of various sizes. Tables 1 and 2, respectively, show these values for five different sets of the matched charts. Note that as all the charts in a set use the same values of *L* and *n*, their ANSS values will be the same. Hence, ANSS is not a relevant measure to compare the statistical performances of the charts. Computations of the SSATS and ANSW values indicate the following facts in general.

If the warning limits of a VSIWL $\bar{X}$ chart are chosen such that its SSATS values for the large shifts match that of a VSI $\bar{X}$ chart then for the small to moderate shifts, its SSATS values are slightly smaller than that of the VSI $\bar{X}$ chart and are similar to that of a VSI (1, 3) $\bar{X}$ chart. Further, the ANSW values of a VSIWL $\bar{X}$ chart are significantly smaller than that of the VSI and VSI (1, 3) $\bar{X}$ charts.

On the other hand, if the warning limits of a VSIWL $\bar{X}$ chart are chosen such that its SSATS values for the large shift are very slightly larger than that of a VSI $\bar{X}$ chart then for the small to moderate shifts, its SSATS values are significantly smaller than that of the VSI and VSI (1, 3) $\bar{X}$ charts and are similar to that of a VSI (2, 3) $\bar{X}$ chart. Besides, its ANSW values are dramatically smaller than that of the other charts and are about 5 to 15% of that of a VSI $\bar{X}$ chart.

Example

The statistically matched VSIWL, VSI (1, 3), VSI (2, 3), and VSI $\bar{X}$ charts with the design parameters, viz., *n*= 4, t_1 = 1.5 hours, t_2 = 0.5 hour, and *L* = 3 are implemented simultaneously and independently for a process. The process is initially in control when the implementation of the charts is started and a shift of size 0.75σ occurs at 5 hours after that. Table 3 shows the sample means taken for the two VSIWL $\bar{X}$ charts along with the corresponding times, sampling interval lengths, and the warning limits used. Table 4 shows the same for VSI (1, 3), VSI (2, 3), and VSI $\bar{X}$ charts. The pair (t_2, w_2) is used for the first trials for the VSIWL charts and the pairs for the subsequent trials are chosen according to the rule of the charts. Similarly, sampling interval length t_2 is used for the first trial for the VSI chart and for the first three trials for the VSI (1, 3) and VSI (2, 3) charts. Sampling interval lengths for the subsequent trials for these charts are chosen according to the respective rules of the charts. Table 5 shows the performances of the five charts which clearly demonstrate the superiority of the VSIWL charts.

Table 3. The details of the VSIWL $\bar{X}$ charts for the process in the example.

VSIWL with w_1 = 1.4, w_2 = 0.2			VSIWL with w_1 = 1.8, w_2 = 0.09		
Time in hours	(*t*(*i*), *w*(*i*))	Zs	Time in hours	(*t*(*i*), *w*(*i*))	Zs
0.5	(0.5, 0.2)	-0.42	0.5	(0.5, 0.09)	1.01
1	(0.5, 0.2)	-1.35	1	(0.5, 0.09)	0.14
1.5	(0.5, 0.2)	1.10	1.5	(0.5, 0.09)	0.50
2	(0.5, 0.2)	-1.43	2	(0.5, 0.09)	2.25
2.5	(0.5, 0.2)	1.33	2.5	(0.5, 0.09)	0.80
3	(0.5, 0.2)	0.46	3	(0.5, 0.09)	0.51
3.5	(0.5, 0.2)	-0.49	3.5	(0.5, 0.09)	-0.66
4	(0.5, 0.2)	0.36	4	(0.5, 0.09)	2.88
4.5	(0.5, 0.2)	0.97	4.5	(0.5, 0.09)	-0.83
5	(0.5, 0.2)	1.88	5	(0.5, 0.09)	2.28
5.5	(0.5, 0.2)	1.38	5.5	(0.5, 0.09)	2.82
6	(0.5, 0.2)	2.46	6	(0.5, 0.09)	3.26
6.5	(0.5, 0.2)	1.48			
7	(0.5, 0.2)	0.61			
7.5	(0.5, 0.2)	2.37			
8	(0.5, 0.2)	1.65			
8.5	(0.5, 0.2)	1.26			
9	(0.5, 0.2)	3.47			

Table 4.The details of the VSI (1, 3), VSI (2, 3), and VSI $\bar{X}$ charts for the process in the example.

VSI (1, 3) with *w* =1.26			VSI (2, 3) with *w* = 0.45			VSI with *w* = 0.67		
Time in hours	*t*(*i*)	Zs	Time in hours	*t*(*i*)	Zs	Time in hours	*t*(*i*)	Zs
0.5	0.5	0.16	0.5	0.5	-0.80	0.5	0.5	-1.08
1	0.5	-0.16	1	0.5	0.72	1	0.5	1.04
1.5	0.5	-0.51	1.5	0.5	-0.66	1.5	0.5	-0.64
3	1.5	-1.09	2	0.5	1.36	3	1.5	2.30
4.5	1.5	-0.22	2.5	0.5	-0.97	3.5	0.5	-0.11
			3	0.5	-0.20			
6	1.5	0.85	4.5	1.5	-0.15	4	1.5	1.33
7.5	1.5	0.91				4.5	0.5	1.42
9	1.5	1.53	6	1.5	2.08	5	0.5	1.18
9.5	0.5	1.70	7.5	1.5	3.13	5.5	0.5	-0.54
10	0.5	3.59				7	1.5	2.35
						7.5	0.5	0.80
						8	0.5	0.58
						9.5	1.5	2.37
						10	0.5	0.72
						10.5	0.5	1.23
						11	0.5	3.08

Table 5. Performances of the charts in the example.

Chart	Time from the shift to the signal	Number of switches during in-control period	Total number of switches until the signal
VSIWL with w_1 = 1.4, w_2 = 0.2	4 hours	0	0
VSIWL with w_1 = 1.8, w_2 = 0.09	1 hour	0	0
VSI (1, 3)	5 hours	1	2
VSI (2, 3)	2.5 hours	1	1
VSI	6 hours	2	8

Conclusion

The idea of variable warning limits is introduced for VSI $\bar{X}$ charts. Expressions for the performance measures, viz., SSATS, ANSS and ANSW for VSIWL $\bar{X}$ charts are developed. The methods presented are general and can be applied to other Shewhart control charts. The effects of variable warning limits on the performances of the charts are evaluated by comparing the performances of VSIWL $\bar{X}$ charts with that of VSI $\bar{X}$ charts with and without runs rules for switching between sampling interval lengths. It is observed that the variable warning limits dramatically reduce the ANSW values of the charts. The idea is even superior to that of runs rules for switching between sampling interval lengths for reducing the ANSW values. Also, it significantly reduces the SSATS values of the charts in detecting small to moderate shifts in the process mean without significantly affecting that in detecting large shifts. It would be interesting to study the application of variable warning limits to the other adaptive control charts.

References

1. Amin, R.W. and Hemasinha, R. 1993. The switching behavior of $\bar{X}$ charts with variable sampling intervals. *Commun. Stat. Theory Meth.*22: 2081-2102.
2. Amin, R.W. and Letsinger, W.C. 1991. Improved switching rules in control procedures using variable sampling intervals. *Commun. Stat. Simul.* 20: 205-230.
3. Amin, R.W. and Widmaier, O. 1999. Sign control charts with variable sampling intervals. *Commun. Stat. Theory Meth.* 28: 1961-1985.

4. Aparasi, F. and Haro, C. 2003. A comparison of T^2 control charts with variable sampling schemes as opposed to MEWMA chart. *Int. J. Prod. Res.* 41: 2169-2182.
5. Brook, D. and Evans, D.A. 1972. An approach to the probability distribution of CUSUM run length. *Biometrik.* 59: 539-549.
6. Celano, G. 2009. Robust design of adaptive control charts for manual manufacturing/inspection workstations. *J. Appl. Stat.* 36: 181-203.
7. Celano, G., Costa, A., and Fichera, S. 2006. Statistical design of variable sample size and sampling interval $\bar{X}$ control charts with run rules. *Int. J. Adv. Manufac. Tech.* 28: 966-977.
8. Chen, Y.K. 2007. Adaptive sampling enhancement for Hotelling's T^2 charts. *Eur. J. Oper. Res.* 178: 841-857.
9. Chen, Y.K., Liao, H.C. and Chang, H.H. 2011. Re-evaluation of adaptive $\bar{X}$ control charts: A cost-effectiveness perspective. *Int. J. Innov. Comp. Info. Cont.* 7: 1229-1242.
10. Costa, A.F.B. 1994. $\bar{X}$ charts with variable sample size. *J. Qual. Technol.* 26: 155-163.
11. Costa, A.F.B. 1999b. Joint $\bar{X}$ and *R* charts with variable sample sizes and sampling intervals. *J. Qual. Technol.* 31: 387-397.
12. Costa, A.F.B. and Rahim, M.A. 2001. Economic design of $\bar{X}$ charts with variable parameters: The Markov chain approach. *J. Appl. Stat.* 28: 875-885.
13. Costa, A.F.B.1999a. $\bar{X}$ charts with variable parameters. *J. Qual. Technol.* 31: 408-416.
14. Dai, Y., Luo, Y., Li, Z., and Wang, Z.2011. A new adaptive CUSUM control chart for detecting the multivariate process mean. *Qual. Reliab. Engg. Int.* 27: 877-884.
15. De Magalhaes, M.S., Costa, A.F.B. and MouraNeto, F.D. 2009. A hierarchy of adaptive $\bar{X}$ control charts. *Int. J. Prod. Econ.*119: 271-283.
16. Epprecht, E.K., Costa, A.F.B. and Mendes, F.C.T. 2003. Adaptive control charts for attributes. *IIE Trans.*35: 567-582.
17. Epprecht, E.K., Simoes, B.F.T. and Mendes, F.C.T. 2010. A variable sampling interval EWMA chart for attributes. *Int. J. Adv. Manufac. Tech.* 49: 281-292.
18. Faraz, A. and Moghadam, M. B. 2009. Hotelling's T^2 control chart with two adaptive sample sizes. *Qual. Quant.* 43: 903-912.
19. Faraz, A. and Saniga, E. 2011.A unification and some corrections to Markov chain approaches to develop variable ratio sampling scheme control charts. *Stat Pap.* 52: 799-811.
20. Jensen, W.A., Bryce, G.R. and Reynolds, M.R. 2008. Design issues for adaptive control charts. *Qual. Reliab. Engg. Int.* 24: 429-445.
21. Jiang, W., Shu, L., and Apley, D. 2008.Adaptive CUSUM procedures with EWMA-based shift estimators. *IIE Trans*. 40: 992-1003.
22. Kooli, I. and Limam, M. 2011. Economic design of an attribute *np* control chart using a variable sample size. *Sequential Anal.* 30: 145-159.
23. Lee, P.H. 2011. Adaptive R charts with variable parameters. *Comput. Stat. Data Anal.* 55: 2003-2010.
24. Li, Z. and Wang, Z. 2010. Adaptive CUSUM of Q chart. *Int. J. Prod. Res.* 48: 1287-1301.
25. Luo, Y., Li, Z. and Wang, Z. 2009. Adaptive CUSUM control chart with variable sampling intervals. *Comput. Stat. Data Anal*. 53: 2693-2701.
26. Mahadik, S.B. 2012a. Exact results for variable sampling interval Shewhart control charts with runs rules for switching between sampling interval lengths. *Commun. Stat. Theory Meth*. 41: 4453-4469.
27. Mahadik, S.B. 2012b. Variable sampling interval Hotelling's T^2 charts with runs rules for switching between sampling interval lengths. *Qual. Reliab. Engg. Int*. 28: 131-140.
28. Mahadik, S.B. 2013. Variable sample size and sampling interval $\bar{X}$ charts with runs rules for switching between sample sizes and sampling interval lengths. *Qual. Reliab. Engg. Int.* 29: 63-76.
29. Mahadik, S.B. and Shirke, D.T. 2007a.On superiority of a variable sampling interval control chart. *J. Appl. Stat.* 34: 443-458.
30. Mahadik, S.B. and Shirke, D.T. 2007b. Economic design of a modified variable sample size and sampling interval $\bar{X}$ chart. *Econ. Qual. Control*. 22: 273-293.
31. Mahadik, S.B. and Shirke, D.T. 2009. A special variable sample size and sampling interval $\bar{X}$ chart. *Commun. Stat. Theory.* 38: 1284-1299.
32. Mahadik, S.B. and Shirke, D.T. 2011. A special variable sample size and sampling interval Hotelling's T^2 chart. *Int. J. Adv. Manufac. Tech.* 53: 379-384.
33. Nenes, G. 2011. A new approach for the economic design of fully adaptive control charts. *Int. J. Prod. Econ.* 131: 631-642.
34. Prabhu, S.S., Montgomery, D.C. and Runger, G.C. 1994. A combined adaptive sample size and sampling interval $\bar{X}$ control scheme. *J. Qual. Technol.* 26: 164-176.
35. Prabhu, S.S., Runger, G.C. and Keats, J.B. 1993. An adaptive sample size $\bar{X}$ chart. *Int. J. Prod. Res.* 31: 2895-2909.
36. Reynolds, M.R., Jr. and Stoumbos, Z.G. 2001. Monitoring the process mean and variance using individual observations and variable sampling intervals. *J. Qual. Technol.* 33: 181-205.
37. Reynolds, M.R., Jr., Amin, R.W., Arnold, J.C. and Nachlas, J.A. 1988. $\bar{X}$ charts with variable sampling interval. *Technometric.* 30: 181-192.
38. Shi, L., Zou, C., Wang, Z., and Kapur, K. 2009. A new variable sampling control scheme at fixed times for monitoring the process dispersion. *Qual. Reliab. Engg. Int.* 25: 961-972.
39. Shu, L., Jiang, W. and Yeung, H.F. 2010.An adaptive CUSUM procedure for signaling process variance changes of unknown sizes. *J. Qual. Technol.* 42: 69-85.
40. Tagaras, G. 1998. A survey of recent developments in the design of adaptive control charts. *J. Qual. Technol.* 30: 212-231.
41. Wu, Z., Tian, Y. and Zhang, S. 2005. Adjusted-loss-function charts with variable sample sizes and sampling intervals. *J. Appl. Stat.* 32: 221-242.
42. Wu, Z., Wang, P. and Wang, Q. 2009. A loss function-based adaptive control chart for monitoring the process mean and variance. *Int. J. Adv. Manufac. Tech*. 40: 948-959.
43. Wu, Z., Zhang, S. and Wang, P.H. 2007. A CUSUM scheme with variable sample sizes and sampling intervals for monitoring the process mean and variance. *Qual. Reliab. Engg. Int.* 23: 157-170.
44. Yang, S.F. and Su, H.C. 2007. Adaptive sampling interval cause-selecting control charts. *Int. J. Adv. Manufac. Tech*. 31: 1169-1180.
45. Yu, F.J. and Hou, J.L. 2006. Optimization of design parameters for $\bar{X}$ control charts with multiple assignable causes. *J. Appl. Stat*. 33: 279-290.
46. Zimmer, L.S., Montgomery, D. C. and Runger, G.C. 1998. Evaluation of a three-state adaptive sample size $\bar{X}$ control chart. *Int. J. Prod. Res.* 36: 733-743.